

Nash equilibria in electric vehicle charging control games: Decentralized computation and connection with social optima[★]

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Abstract

We consider the problem of optimal charging of plug-in electric vehicles (PEVs). We treat this problem as a multi-agent game, where each vehicle/agent is subject to possibly different constraints. Under this set-up, we show that, for any finite number of possibly heterogeneous agents, the PEV charging control game admits a unique Nash equilibrium, which is the optimizer of an auxiliary minimization program. To characterize the population of heterogeneous PEVs as its size grows to infinity, we assume that the parameters defining the constraints of each vehicle are drawn randomly from a given distribution. We are then able to show that, as the number of agents tends to infinity, the value of the game and the social optimum of the cooperative counterpart of the problem under study coincide for almost any choice of the random heterogeneity parameters. Moreover, in the case of a discrete probability distribution, we provide a systematic way to abstract agents in homogeneous groups and show that the effect of heterogeneity averages out as their number tends to infinity. We also show that, for any finite number of agents, the desired Nash equilibrium can be computed exactly by means of a regularized Jacobi algorithm, and support our theoretical results via a detailed simulation study.

Key words: Decentralized optimization, mean field games, optimal charging control, electric vehicles, Jacobi algorithm.

1 Introduction

Electric vehicles obtain some or all of their energy from the electricity grid, and are typically referred to as plug-in electric vehicles (PEVs). Their penetration is expected to increase significantly in the near future, since, not only they contribute to pollution reduction, but, by appropriately scheduling their charging status (e.g., charging over low demand/electricity price periods), they also serve as virtual dynamic storage, contributing to the stability of the electric grid (see [3, 4, 16, 22] and references therein).

The price of electricity is actually affected by the demand, and if we look at the PEV charging problem from

a social welfare perspective, a centralized solution would be preferable, since it would minimize the global cost. However, optimal PEV charging becomes more challenging as the vehicles population size grows [15]. This is due to the fact that centralized computation of the vehicles charging strategies becomes prohibitive both from a computational and a communication point of view, since all vehicles should provide information, e.g., on their charging status and constraints on the charging rate. Irrespectively of the population size, vehicles are often not willing to share information that they consider private with other vehicles, and typically act as selfish agents that seek to minimize their local charging energy cost, without being concerned with social welfare paradigms. This gives rise to multi-agent non-cooperative games, and the main concern is then the characterization and computation of Nash equilibrium strategies associated with such games.

To this purpose, research activity has been concentrated towards the development of decentralized algorithmic methodologies. The PEV charging control problem can in fact be naturally formulated as a decentralized control problem where each vehicle/agent solves a local optimization problem, minimizing its own charging energy cost and calculating a tentative solution, which is then

[★] Research was supported by the European Commission, H2020, under the project UnCoVerCPS, grant number 643921.

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broadcasted to a central authority. On the basis of the information received, this authority sends then an aggregate price signal to all agents, which then respond to this price driver. Aggregative and mean-field game theory offers an already mature mathematical machinery for decentralized computation of Nash equilibria in multi-agent games. A complete theoretical characterization is provided in [12, 14] for stochastic continuous-time problems where, however, agents are not subject to constraints. The deterministic, discrete-time problem variant, was investigated in [18], and was further extended in [21] to account for the presence of constraints. Moreover, the authors of [11] provided a range of iterative decentralized PEV charging schemes accompanied with conditions that characterize their convergence properties derived using fixed-point theoretic tools [1].

One challenge associated with the aforementioned references is that, for any finite number of agents, the constructed methodology results only in approximate Nash equilibria; an exact Nash equilibrium is reached only in the limiting case where the number of agents tends to infinity. The recent work of [20], provides a gradient based methodology to overcome this issue and, using tools from variational inequalities [7], shows convergence to a Nash equilibrium for a finite number of agents. This is achieved under the assumption that vehicles are price anticipating agents, being aware of the way the total consumption of the PEV population affects the price that drives the behavior of individual vehicles. Under the same assumption, [5, 9] propose a regularized version of each agent local problem, but treat the problem from a cooperative perspective, showing convergence to a social welfare optimum. In [6] this cooperative algorithm is further extended to account for general, convex pay-off functions.

A second challenge in existing PEV charging control algorithms is that there is no common awareness on how the resulting Nash equilibrium solution is related to the associated social welfare optimum, had the PEVs been acting in a cooperative manner, and how this is affected by heterogeneity of vehicles. A partial answer to the first question was given in [18] for the case of a homogeneous population of PEVs, that are not subject to constraints. A more general treatment of the problem is proposed in the recent work of [17], where the authors show equivalence of Nash equilibria and social optima at the limiting case of infinite agent populations. This is achieved by means of a primal-dual analysis, which for the case where the number of agents is finite results in approximate and not exact Nash equilibria.

In this paper we consider the problem of PEV charging control, where each vehicle is subject to possibly different constraints. We represent constraint heterogeneity by assuming that the parameters defining the constraints of each vehicle are drawn randomly from a given distribution. Under this set-up, we are able to address both

challenges that appear in existing approaches in the literature.

More specifically, our paper provides the following contributions:

- (1) For any finite number of possibly heterogeneous agents, we show that the PEV charging control game admits a unique Nash equilibrium, which is the minimizer of an auxiliary minimization program (Proposition 3). The construction of such a problem is related to the fact that the underlying game is potential, hence its structure follows the arguments of [8], however, our proof line is different and is based on fixed-point theoretic results. We further prove that as the number of agents tends to infinity the value of the game achieved by the Nash equilibrium and the social optimum of the cooperative counterpart of the problem under study coincide for almost any choice of the random heterogeneity parameters (Theorem 2). This result extends [18] to the case of heterogeneous agents that are subject to constraints, without resorting to approximate Nash equilibria as in [17], and following a fundamentally different analysis that does not require primal-dual update steps.
- (2) We show that if the distribution of the random parameters that render agents' constraints heterogeneous is discrete, agents can be abstracted in homogeneous groups, thus giving rise to an equivalent problem with fewer decision variables (Proposition 4). Based on this result, we further show that for almost any realization of the random heterogeneity parameters, as the number of agents tends to infinity, the value of the game (and hence the social optimum based on the discussion above) tends to a deterministic quantity (Theorem 3). This implies that in the limiting case of populations with infinite size the effect of heterogeneity averages out. In contrast to [11, 17, 20] we provide a probabilistic treatment to analyze the asymptotic effect of agents' heterogeneity. Our analysis then serves as the discrete time counterpart of the approach in [12], where heterogeneity is modeled probabilistically in a continuous time setting.
- (3) In view of the equivalence between Nash equilibria and social minimizers of an auxiliary problem, we show that the regularized Jacobi algorithm, constructed in our earlier work for decentralized optimization [6], is also applicable for decentralized Nash equilibrium computations. In contrast to [11, 17], we show convergence of this algorithm (Theorem 4) to the exact Nash equilibrium for any finite number of agents. This same result was recently shown in [20]. However, we follow a fundamentally different proof line from [20], which, apart from connecting Nash equilibria with social optima, allows us to characterize Nash equilibria as fixed-points of certain mappings for more general games than the

PEV charging one, possibly with non-differentiable pay-off functions (Corollary 1). This latter property is not captured by [20], where a gradient based algorithm is used instead of a regularized Jacobi algorithm.

The remainder of the paper is organized as follows: Section 2 introduces the non-cooperative PEV charging control game under study, along with its social welfare optimization counterpart. Section 3 shows that, for any finite number of possibly heterogeneous agents, the associated game admits a unique Nash equilibrium, which is the social optimum of an auxiliary minimization program. Moreover, as the number of agents tends to infinity, the value of the game and the social welfare optimum of the original problem tend to coincide. In Section 4, we show that agents can be abstracted in homogeneous groups, and when the number of agents tends to infinity the effect of heterogeneity averages out. Section 5 provides a decentralized algorithm to compute exact Nash equilibria for any finite number of agents, and performs a simulation study to support our theoretical analysis. Finally, Section 6 concludes the paper and provides some directions for future work.

2 Electric vehicle charging control problem

2.1 Cooperative set-up

We first consider the case of m PEVs that seek to determine their charging profile along some discrete time horizon $[0, h - 1]$ of arbitrary length $h \in \mathbb{N}$ so as to minimize the total charging cost for the entire fleet. This corresponds to a cooperative set-up that is likely to occur when vehicles belong to the same managing entity. We outline here the formulation of this problem since it will be compared against the non-cooperative charging strategy of Section 2.2. To this end, let $H = \{0, 1, \dots, h - 1\}$ and $I = \{0, 1, \dots, m\}$ be the time and agent index sets, respectively. Note that in the set I we have $m + 1$ agent indices instead of m . The agent indexed by 0 is in fact virtual and introduced to represent some additional fixed demand besides the one requested by the PEVs. This choice allows us to avoid cluttering notation in the subsequent derivations.

Consider the following optimization program:

$$\min_{\{x^{it} \in \mathbb{R}\}_{t \in H, i \in I}} \sum_{i \in I} \sum_{t \in H} x^{it} \left(p^t \sum_{j \in I} x^{jt} \right) \quad (1)$$

subject to

$$\sum_{t \in H} x^{it} = \gamma^i, \text{ for all } i \in I, \quad (2)$$

$$x^{it} \in [\underline{x}^{it}, \bar{x}^{it}], \text{ for all } t \in H, i \in I, \quad (3)$$

where $x^{it} \in \mathbb{R}$ is the charging rate of vehicle i at time t , and $p^t > 0$ is an electricity price coefficient at time t .

The price of electricity is given by $p^t \sum_{j \in I} x^{jt}$, and is assumed to depend linearly on the total demand through p^t . The linear dependency of price with respect to the total demand models the fact that price depends on demand, and, agents/vehicles are price anticipating authorities, anticipating their consumption to have an effect on the electricity price (see [10] for further elaboration). Dependency of price on the PEV demand is affine due the presence of x^{0t} .

The objective function in (1) encodes the total electricity cost over $[0, h - 1]$. Constraint (2) represents a prescribed charging level $\gamma^i \in \mathbb{R}$, $\gamma^i > 0$, to be reached by each vehicle i at the end of the considered time horizon H , whereas (3) imposes minimum ($\underline{x}^{it} \in \mathbb{R}$, $\underline{x}^{it} \geq 0$) and maximum ($\bar{x}^{it} \in \mathbb{R}$, $\bar{x}^{it} < \infty$) limits, respectively, on x^{it} . By appropriately choosing \underline{x}^{0t} , and setting $\bar{x}^{0t} = \underline{x}^{0t}$ and $\gamma^0 = \sum_{t \in H} \underline{x}^{0t}$, the charging strategy of the virtual agent 0 can match any given non-PEV demand profile.

For all $i \in I$, let $x^i = [x^{i0}, \dots, x^{i(h-1)}]^\top \in \mathbb{R}^{|H|}$, where $|\cdot|$ denotes the cardinality of its argument. Let also $f : \mathbb{R}^{|H|} \times \mathbb{R}^{(m+1)|H|} \rightarrow \mathbb{R}$ be such that, for all $i \in I$, for any $(x^i, x^{-i}) \in \mathbb{R}^{(m+1)|H|}$,

$$f(x^i, x^{-i}) = \sum_{t \in H} x^{it} \left(p^t \sum_{\substack{j \in I \\ j \neq i}} x^{jt} + p^t x^{it} \right), \quad (4)$$

where by the notation x^{-i} we imply a vector including the decision variables of all vehicles except vehicle i . Moreover, for all $i \in I$, let

$$X^i = \{x^i \in \mathbb{R}^{|H|} : \sum_{t \in H} x^{it} = \gamma^i \text{ and } x^{it} \in [\underline{x}^{it}, \bar{x}^{it}], \text{ for all } t \in H\}, \quad (5)$$

denote the constraint set corresponding to vehicle i . Based on this notation, we shall rewrite (1)-(3) in the following more compact form:

$$\mathcal{P} : \min_{\{x^i \in X^i\}_{i \in I}} \sum_{i \in I} f(x^i, x^{-i}). \quad (6)$$

and refer to its optimal solution as social optimum. Note that local utility functions, e.g., $g^i(x^i)$, that depend only on the decision vector x^i of each vehicle i , $i \in I$, and are possibly different per vehicle, can be incorporated in \mathcal{P} . This can be achieved by an epigraphic reformulation of these local utility functions using auxiliary decision variables μ^i , $i \in I$. In that case the objective function in (6) will be $\sum_{i \in I} f(x^i, x^{-i}) + \mu^i$, while the constraints $g^i(x^i) \leq \mu^i$ would be embedded in the constraint set X^i , $i \in I$, (see also [6] for more details). That way, even if we use the same pay-off function for all vehicles in (6), we account for different limits or preferences per vehicle.

We impose the following standing assumption to ensure feasibility of \mathcal{P} .

Assumption 1 Fix any $m \geq 1$. Let $\gamma^i > 0, i \in I \setminus \{0\}$, and $\gamma^0 \geq 0$. We assume that the constraint sets $X^i, i \in I$, are nonempty and compact. Moreover, the electricity price coefficient satisfies $p^t > 0$, for all $t \in H$.

Note that γ^0 is allowed to be zero to encode the case where there is no non-PEV demand. The second part of Assumption 1 is only needed for the proof of Theorem 2, but is naturally satisfied in situations of practical relevance.

Denote the set of social optima M of \mathcal{P} by

$$M = \arg \min_{\{x^i \in X^i\}_{i \in I}} \sum_{i \in I} f(x^i, x^{-i}). \quad (7)$$

Note that (7) involves minimizing a continuous function (as an effect of being convex), over a compact set (due to Assumption 1). As such, the minimum is achieved due to Weierstrass' theorem in [2, Proposition A.8, p. 625]. Under a similar reasoning all other minimization problems defined in the sequel are well defined.

More precisely, problem \mathcal{P} involves a convex, quadratic minimization program. Algorithm 1 in [6] offers a decentralized approach to compute a minimizer to this problem that belongs to M .

2.2 Non-cooperative set-up

We now consider the case where the m vehicles act in an non-cooperative manner. In particular, each vehicle/agent $i, i \in I$, aims at determining a charging profile x^i that minimizes its pay-off function $f(x^i, x^{-i})$, as this is given by (4), which depends on its own decision vector x^i and on the other agents decision vector x^{-i} , subject to a local constraint $x^i \in X^i$, where X^i is defined in (5).

This non-cooperative behavior naturally gives rise to a gaming setting. We say that for all $i, i \in I$, the tuple (x^i, x^{-i}) is a Nash equilibrium of the game, if each agent i , given the strategies x^{-i} of the other agents, has no interest in changing its own strategy x^i . In other words, unilateral deviations in the agents' local strategies can not lead to an improvement in their pay-offs. This is formally stated in the following definition.

Definition 1 Consider a non-cooperative game where each agent i has a pay-off function $f(\cdot, x^{-i})$ and a constraint set $X^i, i \in I$. The set of Nash equilibria N of the game is given by

$$N = \{x \in X : f(x^i, x^{-i}) \leq f(\zeta^i, x^{-i}) \text{ for all } \zeta^i \in X^i, i \in I\}, \quad (8)$$

where $x = (x^0, \dots, x^m)$ and $X = X^0 \times \dots \times X^m$.

Since each agent has a pay-off function of the same structure, the resulting game is a potential game [8, 25]. In the next section we show that set of Nash equilibria N coincides with the set of optima of an auxiliary cooperative optimization program, which is albeit different from the one of Section 2.1. We show however, that in the limiting case of an infinite population of agents, this auxiliary problem tends to the one of Section 2.1, thus establishing an equivalence result between the set of social optima M and the set of Nash equilibria N , as far as the value of the game is concerned. Based on the results of Section 3, we then provide in Section 5.1 a decentralized algorithm that results in an element of N , i.e., a Nash equilibrium of the game of interest.

3 Nash equilibria versus social optima

3.1 Nash equilibria as fixed-points

The results of this subsection do not require the pay-off function to exhibit the form of (4) and are more general. In fact, they hold also for the case where every agent has a different pay-off function, which has to be convex with respect to the decision vector of the particular agent (with the other decision vectors being fixed), but is allowed to be non-differentiable.

For each $i, i \in I$, consider the mappings $T^i : X \rightarrow X^i$ and $\tilde{T}^i : X \rightarrow X^i$, defined such that, for any $x \in X$,

$$T^i(x) = \arg \min_{z^i \in X^i} \|z^i - x^i\|^2 \quad (9)$$

subject to

$$f(z^i, x^{-i}) \leq \min_{\zeta^i \in X^i} f(\zeta^i, x^{-i}),$$

$$\tilde{T}^i(x) = \arg \min_{z^i \in X^i} f(z^i, x^{-i}) + c\|z^i - x^i\|^2, \quad (10)$$

for any $c > 0$. Note that both mappings are well defined since both the minimizers of (9) and (10) are unique. As for the mapping in (9), a tie-break rule is implemented to select, in case $f(\cdot, x^{-i})$ admits multiple minimizers over X^i , the one closer to x^i with respect to the Euclidean norm. In contrast, the mapping \tilde{T}^i in (10) includes in the objective function an additional term weighted by $c > 0$, which penalizes the deviations from the current decision vector x^i and makes it strictly convex. Notice, with a slight abuse of notation, by $T^i(x)$ and $\tilde{T}^i(x)$, we imply the minimizers of (9) and (10), respectively, and not the corresponding (singleton due to uniqueness) sets.

Define also the mappings $T : X \rightarrow X$ and $\tilde{T} : X \rightarrow X$, such that their components are given by T^i and \tilde{T}^i , respectively, for $i \in I$, i.e., $T = (T^0, \dots, T^m)$ and $\tilde{T} = (\tilde{T}^0, \dots, \tilde{T}^m)$. The mappings T and \tilde{T} can be equiva-

lently written as

$$T(x) = \arg \min_{z \in X} \sum_{i \in I} \|z^i - x^i\|^2 \quad (11)$$

subject to

$$f(z^i, x^{-i}) \leq \min_{\zeta^i \in X^i} f(\zeta^i, x^{-i}), \quad \forall i \in I,$$

$$\tilde{T}(x) = \arg \min_{z \in X} \sum_{i \in I} [f(z^i, x^{-i}) + c\|z^i - x^i\|^2], \quad (12)$$

given that the terms inside the summation in (11) and (12) are decoupled.

The set of fixed points for the mappings T and \tilde{T} is, respectively, given by

$$F_T = \{x \in X : x = T(x)\} \quad (13)$$

$$F_{\tilde{T}} = \{x \in X : x = \tilde{T}(x)\}. \quad (14)$$

We start by showing that the set of Nash equilibria N and the set of fixed-points F_T of the mapping T in (11) coincide. This is summarized in the following proposition.

Proposition 1 *Under Assumption 1, $N = F_T$.*

Proof 1) $N \subseteq F_T$: Fix any $x \in N$. For each $i \in I$, denote x by (x^i, x^{-i}) . The fact that $x \in N$ implies that x^i is a minimizer of $f(\cdot, x^{-i})$, for all $i \in I$, indeed according to (8), $f(x^i, x^{-i})$ will be no greater than the values that f may take if evaluated at (ζ^i, x^{-i}) , for any $\zeta^i \in X^i$, i.e., $f(x^i, x^{-i}) \leq f(\zeta^i, x^{-i})$, for all $\zeta^i \in X^i$. The last statement can be equivalently written as $f(x^i, x^{-i}) \leq \min_{\zeta^i \in X^i} f(\zeta^i, x^{-i})$ which means that x satisfies the inequality in (11). Moreover, x is also optimal for the objective function in (11), since it results in zero cost. Hence, $x = T(x)$, which by (13) implies that $x \in F_T$, thus concluding the first part of the proof.

2) $F_T \subseteq N$: Fix any $x \in F_T$. By the definition of F_T , and due to the inequality in (11) that is embedded in the definition of T , we have that for all $i \in I$, $f(x^i, x^{-i}) \leq \min_{\zeta^i \in X^i} f(\zeta^i, x^{-i})$. The last statement implies that x^i is a minimizer of $f(\cdot, x^{-i})$ over X^i , and hence $f(x^i, x^{-i}) \leq f(\zeta^i, x^{-i})$, $\forall \zeta^i \in X^i$, $\forall i \in I$, which due to (8) implies that $x \in N$, thus concluding the second part of the proof. \square

We next show that the set of fixed-points F_T of T in (13) and the set of fixed-points $F_{\tilde{T}}$ of \tilde{T} in (14) coincide. This is summarized in the following proposition.

Proposition 2 *Under Assumption 1, $F_T = F_{\tilde{T}}$.*

Proof 1) $F_T \subseteq F_{\tilde{T}}$: Fix any $x \in F_T$. By the definition of F_T , and due to the inequality in (11) that is embedded in the definition of T , we have that for all $i \in I$, $f(x^i, x^{-i}) \leq$

$\min_{\zeta^i \in X^i} f(\zeta^i, x^{-i})$. The last statement implies that x^i is a minimizer of $f(\cdot, x^{-i})$ over X^i , and hence $f(x^i, x^{-i}) \leq f(\zeta^i, x^{-i})$, $\forall \zeta^i \in X^i$, $\forall i \in I$. Therefore, we would also have that $f(x^i, x^{-i}) \leq f(\zeta^i, x^{-i}) + c\|\zeta^i - x^i\|^2$, $\forall \zeta^i \in X^i$, $\forall i \in I$. The latter, due to (10) implies that $x^i = T^i(x)$, for all $i \in I$, i.e., $x \in F_{\tilde{T}}$, thus concluding the first part of the proof.

2) $F_{\tilde{T}} \subseteq F_T$: Fix any $x \in F_{\tilde{T}}$. By the definition of $F_{\tilde{T}}$, and due to (10), the latter implies that $x^i = \tilde{T}^i(x)$ for all $i \in I$. We thus have that, for all $i \in I$,

$$f(x^i, x^{-i}) \leq f(\zeta^i, x^{-i}) + c\|\zeta^i - x^i\|^2, \quad \text{for all } \zeta^i \in X^i. \quad (15)$$

If in addition x^i minimizes $f(\cdot, x^{-i})$ over X^i , for all $i \in I$, then x^i would satisfy the inequality in (9), while it results in zero cost. We would thus have that $x^i = T^i(x)$, for all $i \in I$, and hence $x \in F_T$ (see also the first part of the proof of Proposition 1), concluding the second part of the proof.

To show that, for all $i \in I$, x^i minimizes $f(\cdot, x^{-i})$ over X^i , assume for the sake of contradiction that this is not the case and there exists $z^i \in X^i$, $z^i \neq x^i$, such that $f(z^i, x^{-i}) < f(x^i, x^{-i})$. For any $\alpha \in (0, 1)$, let $\zeta^i = \alpha z^i + (1 - \alpha)x^i$. Note that by convexity of X^i , $\zeta^i \in X^i$, whereas by convexity of $f(\cdot, x^{-i})$ with respect to its first argument we have that

$$f(\zeta^i, x^{-i}) \leq \alpha f(z^i, x^{-i}) + (1 - \alpha)f(x^i, x^{-i}), \quad (16)$$

which, by rearranging some terms, can be rewritten as

$$f(\zeta^i, x^{-i}) + \alpha(f(x^i, x^{-i}) - f(z^i, x^{-i})) \leq f(x^i, x^{-i}). \quad (17)$$

Note that, since $f(x^i, x^{-i}) - f(z^i, x^{-i}) > 0$, $\|z^i - x^i\| > 0$ and $c > 0$, there exists $\alpha \in (0, 1)$ such that

$$\alpha(f(x^i, x^{-i}) - f(z^i, x^{-i})) > c\alpha^2\|z^i - x^i\|^2 = c\|\zeta^i - x^i\|^2, \quad (18)$$

where the equality follows from the definition of ζ^i (note that ζ^i depends on the choice of α). By (17) and (18) we have that there exists α such that

$$f(\zeta^i, x^{-i}) + c\|\zeta^i - x^i\|^2 < f(x^i, x^{-i}). \quad (19)$$

The last statement, together with (15), leads to a contradiction, thus showing that x^i minimizes $f(\cdot, x^{-i})$ over X^i , concluding the proof. \square

An alternative proof for an equivalence result similar to the one of Proposition 2 was provided in [6, Proposition 3], relying, however, on the assumption that the objective functions involved are convex and differentiable. The proof provided here significantly deviates from that in [6, Proposition 3], and holds for convex, but possibly non-differentiable, objective functions.

The following corollary is a direct consequence of Propositions 1 and 2.

Corollary 1 *Under Assumption 1, $N = F_{\tilde{T}}$.*

3.2 Nash equilibria as social optima of an auxiliary problem

We show that the set of Nash equilibria N defined in (8) of the game in Section 2.2 coincides with the set of optimizers of an auxiliary minimization program. To this end, for all $i \in I$, let

$$f_a(x^i) = \sum_{t \in H} p^t (x^{it})^2, \quad (20)$$

and consider the following minimization problem.

$$\mathcal{P}_a : \min_{\{x^i \in X^i\}_{i \in I}} \sum_{i \in I} [f(x^i, x^{-i}) + f_a(x^i)]. \quad (21)$$

We then have the following proposition.

Proposition 3 *Under Assumption 1, the set of Nash equilibria N , and the set of minimizers of \mathcal{P}_a coincide, i.e.,*

$$N = \arg \min_{\{x^i \in X^i\}_{i \in I}} \sum_{i \in I} [f(x^i, x^{-i}) + f_a(x^i)]. \quad (22)$$

Proof Problem \mathcal{P}_a is a centralized convex optimization program. By Corollary 1 of [6], the set of minimizers of \mathcal{P}_a coincides with the set of fixed points of the mapping $\tilde{T}_a = (\tilde{T}_a^0, \dots, \tilde{T}_a^m)$ (see also equation (5) in [6]), where, for all $i \in I$, for any $c > 0$,

$$\begin{aligned} \tilde{T}_a^i(x) = \arg \min_{z^i \in X^i} & f(z^i, x^{-i}) + f_a(z^i) \\ & + \sum_{\substack{k \in I \\ k \neq i}} [f(x^k, (z^i, x^{-\{k,i\}})) + f_a(x^k)] \\ & + 2c \|z^i - x^i\|^2, \end{aligned} \quad (23)$$

where $f(x^k, (z^i, x^{-\{k,i\}})) = \sum_{t \in H} x^{kt} \left(p^t \sum_{\substack{k \in I \\ k \neq i}} x^{kt} + p^t z^{it} \right)$, for all $k \in I$, $k \neq i$, encodes the fact that the decision vector z^i of agent i appears also in the terms with $k \neq i$. By $x^{-\{k,i\}}$ we mean the elements of x but for the

ones corresponding to agents k and i . The interpretation of (23) is that we minimize the objective function in (21) with respect to the decision vector of agent i , where all the other decision vectors are fixed to the values included in vector x .

Therefore, and due to (20), we have that

$$\begin{aligned} \tilde{T}_a^i(x) = \arg \min_{z^i \in X^i} & \left[\sum_{t \in H} z^{it} \left(p^t \sum_{\substack{j \in I \\ j \neq i}} x^{jt} + p^t z^{it} \right) \right. \\ & \left. + \sum_{t \in H} z^{it} p^t \sum_{\substack{k \in I \\ k \neq i}} x^{kt} + \sum_{t \in H} p^t (z^{it})^2 \right] \\ & + 2c \|z^i - x^i\|^2, \end{aligned} \quad (24)$$

where all terms that have been dropped from the objective function in (23) do not depend on the decision vector z^i . Rearranging the terms, we obtain

$$\begin{aligned} \tilde{T}_a^i(x) = \arg \min_{z^i \in X^i} & 2 \sum_{t \in H} z^{it} \left(p^t \sum_{\substack{j \in I \\ j \neq i}} x^{jt} + p^t z^{it} \right) \\ & + 2c \|z^i - x^i\|^2 \\ = \arg \min_{z^i \in X^i} & f(z^i, x^{-i}) + c \|z^i - x^i\|^2 \\ = \tilde{T}^i(x), \end{aligned} \quad (25)$$

where in the second equality we used (4) and rescaled the objective by a factor of 2, since this does not affect the resulting minimizer. The last equality follows from the definition of the mapping \tilde{T}^i in (10).

Equation (25) implies that the mappings \tilde{T}_a^i and \tilde{T}^i are identical, hence, the set of minimizers of \mathcal{P}_a coincides with the set of fixed-points of \tilde{T}^i . However, by Corollary 1, the latter coincides with the set of Nash equilibria N , thus concluding the proof. \square

Note that the objective function in (21) is strictly convex due to the presence of the auxiliary term. Therefore, it admits a unique minimizer and, as a result of Proposition 3, the game of Section 2.2 admits a unique Nash equilibrium. By Corollary 1 this in turn implies that the mapping \tilde{T} has a unique fixed-point. The uniqueness of the Nash equilibrium is due to the equivalence result of (22), which relies on the particular structure of the objective functions in (4); for general convex pay-off functions (22) does not necessarily hold, and as a result N may not be a singleton.

The interpretation of (21) is that the auxiliary term acts like a variance penalty in regularization methods (similar to overfitting prevention in regression algorithms), since it promotes least norm solutions, thus implicitly enforcing uniformity in the agents' decision variables.

As shown in the next section, the relative importance of this term becomes negligible as the number of agents increases.

3.3 Asymptotic equivalence result

In this subsection we show that in the limiting case of an infinite population of agents, the optimal value of \mathcal{P}_a in (21) approaches the one of the cooperative optimization problem \mathcal{P} in (6). Under Proposition 3, this in turn implies that the Nash equilibrium of the game in Section 2.2 achieves the social welfare optimum. In other words, even if agents act in a non-cooperative manner, as their number increases, they tend to a social welfare optimizing behavior.

For our analysis we assume that the price coefficients $\{p^t\}_{t \in H}$ are deterministic known quantities, whereas the consumption level $\gamma^i, i \in I \setminus \{0\}$ in (2) and the upper and lower limits in (3) are random variables, extracted according to a given probability distribution. This setting allows to easily characterize a possibly infinite population of agents. We impose the following assumption on the infinite sequence of random vectors $\{\gamma^i, \underline{x}^i, \bar{x}^i\}_{i \geq 1}$, where $\underline{x}^i = [\underline{x}^{i0}, \dots, \underline{x}^{i(h-1)}]$, $\bar{x}^i = [\bar{x}^{i0}, \dots, \bar{x}^{i(h-1)}]$.

Assumption 2 Let $\{\gamma^i, \underline{x}^i, \bar{x}^i\}_{i \geq 1}$ be an infinite sequence of random vectors on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ ³. We assume that

- (1) $\{\gamma^i, \underline{x}^i, \bar{x}^i\}_{i \geq 1}$ are a sequence of independent and identically distributed (i.i.d.) random vectors.
- (2) $\{\gamma^i\}_{i \geq 1}$ are positive random variables, while $\{\underline{x}^i, \bar{x}^i\}_{i \geq 1}$ are non-negative random vectors.
- (3) For any $i \geq 1$, $\mathbb{E}[\gamma^i] < \infty$ and $\mathbb{E}[(\gamma^i)^2] < \infty$, where $\mathbb{E}[\cdot]$ denotes the expectation operator associated with the probability measure \mathbb{P} .

Note that we do not impose Assumption 2 for the virtual agent indexed by 0, since its demand is a deterministic quantity. As a consequence of the second part of Assumption 2, $\mathbb{E}[\gamma^i] > 0$, for any $i \geq 1$. Recall that in Assumption 1 we require feasibility of \mathcal{P} (and, hence, of \mathcal{P}_a) for a finite number of agents. this entails that the joint probability distribution of $\{\gamma^i, \underline{x}^i, \bar{x}^i\}_{i \geq 1}$ should be such that feasibility is ensured, namely the lower and upper bounds on charging rate must be compatible with the charging level γ^i to be reached. For the subsequent analysis we employ the following law of large numbers type of argument. Note that we will write that an event holds (\mathbb{P} -a.s.) when it holds with probability one with respect to \mathbb{P} .

³ Note that if $\{\gamma^i, \underline{x}^i, \bar{x}^i\}, i \geq 1$, is defined on a given set, by \mathbb{P} we denote the probability measure induced on the infinite cartesian product of these sets. For more details on the mathematical construction of such a measure the reader is referred to [24] (Section 2.4.1, p. 29).

Theorem 1 ([23], Chapter IV, §3, Theorem 3)

Let $\{y^j\}_{j \geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{E}[|y^1|] < \infty$. For any given index set J_m with cardinality $|J_m| = m$, we then have that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j \in J_m} y^j = \mathbb{E}[y^1], \quad (\mathbb{P}\text{-a.s.}) \quad (26)$$

Consider any given index set H with $|H| = h, h \geq 1$, and let $y^t \in \mathbb{R}, y^t \geq 0$, for all $t \in H$. Let also $\bar{y} \in \mathbb{R}$ such that $\sum_{t \in H} y^t = \bar{y}$. Due to norm equivalence we have that $\frac{\|y\|_1}{\sqrt{h}} \leq \|y\|_2 \leq \|y\|_1$, where $y = (y^1, \dots, y^h)$. The latter implies that

$$\frac{\bar{y}^2}{h} \leq \sum_{t \in H} (y^t)^2 \leq \bar{y}^2, \quad (27)$$

which we will exploit in the proof of Theorem 2 below.

Denote by

$$F^m(x) = \sum_{i \in I} f(x^i, x^{-i}) \quad (28)$$

the objective function of \mathcal{P} in (6), and let

$$F_a^m(x) = \sum_{i \in I} f_a(x^i). \quad (29)$$

The objective function of \mathcal{P}_a in (21) can be thus written as $F^m(x) + F_a^m(x)$. We introduce the superscript m in our notation to emphasize the fact that the relevant objective functions correspond to a set-up of m agents, since in the sequel we will let m tend to infinity. Notice that, for any $x \in X$,

$$\begin{aligned} F^m(x) &= \sum_{t \in H} p^t \left(\sum_{i \in I} x^{it} \right)^2 \geq \underline{p} \sum_{t \in H} \left(\sum_{i \in I} x^{it} \right)^2 \\ &\geq \underline{p} \frac{\left(\sum_{i \in I} \gamma^i \right)^2}{h} > 0, \end{aligned} \quad (30)$$

where the first inequality is obtained by setting $\underline{p} = \min_{t \in H} p^t$. To see the second inequality notice that $\sum_{t \in H} \left(\sum_{i \in I} x^{it} \right) = \sum_{i \in I} \left(\sum_{t \in H} x^{it} \right) = \sum_{i \in I} \gamma^i$. The desired inequality follows then by the left-hand side of (27) with $\sum_{i \in I} x^{it}, \sum_{i \in I} \gamma^i$ in place of y^t and \bar{y} , respectively. The last inequality is strict, due to the fact that $\underline{p} > 0$ (H is a finite set) as a result of the second part of Assumption 1, and the fact that $\gamma^i > 0$, for all $i \geq 1$, due to the first part of Assumption 2.

We are now in a position to state the following theorem, which is the main result of this section. We show that the

value obtained by evaluating F^m at the optimal solution of \mathcal{P}_a , which due to Proposition 3 corresponds to the Nash equilibrium for the game of Section 2.2, tends to the social welfare optimum (optimal value of \mathcal{P}) as the number of agents tends to infinity.

Theorem 2 *Consider Assumptions 1 and 2. Let $x^* \in X$, $x_a^* \in X$ be any minimizer of \mathcal{P} and \mathcal{P}_a , respectively. We then have that*

$$\lim_{m \rightarrow \infty} \frac{F^m(x_a^*)}{F^m(x^*)} = 1, \quad (\mathbb{P}\text{-a.s.}), \quad (31)$$

where $F^m(x^*) > 0$.

Proof Let $x, x_a \in X$ be feasible solutions, possibly different, of \mathcal{P} and \mathcal{P}_a , respectively. By the definition of F^m and F_a^m in (28) and (29), and since $F^m(x) > 0$ for any $x \in X$, we have that

$$\frac{F_a^m(x_a)}{F^m(x)} = \frac{\sum_{t \in H} p^t \sum_{i \in I} (x_a^{it})^2}{\sum_{t \in H} p^t \left(\sum_{i \in I} x^{it} \right)^2}. \quad (32)$$

Let $\bar{p} = \max_{t \in H} p^t$ and $\underline{p} = \min_{t \in H} p^t > 0$, where the inequality is strict due to Assumption 1. We then have that

$$\frac{F_a^m(x_a)}{F^m(x)} \leq \frac{\bar{p} \sum_{t \in H} \sum_{i \in I} (x_a^{it})^2}{\underline{p} \sum_{t \in H} \left(\sum_{i \in I} x^{it} \right)^2}. \quad (33)$$

Since x_a^{it} is feasible for \mathcal{P}_a , we have that $\sum_{t \in H} x_a^{it} = \gamma^i$, for all $i \in I$. By the right-hand side of (27) with x^{it} , γ^i in place of y^t and \bar{y} , respectively, we obtain that for all $i \in I$,

$$\sum_{t \in H} (x_a^{it})^2 \leq (\gamma^i)^2. \quad (34)$$

By the derivation of (30), we obtain that

$$\sum_{t \in H} \left(\sum_{i \in I} x^{it} \right)^2 \geq \frac{\left(\sum_{i \in I} \gamma^i \right)^2}{h}. \quad (35)$$

Employing (34), (35), and by exchanging the summation order in the numerator of (33), we have that

$$\frac{F_a^m(x_a)}{F^m(x)} \leq \frac{\bar{p} h \sum_{i \in I} (\gamma^i)^2}{\underline{p} \left(\sum_{i \in I} \gamma^i \right)^2} = \frac{\bar{p} h \sum_{i \in I} (\gamma^i)^2}{\underline{p} m \left(\sum_{i \in I} \gamma^i \right)^2}. \quad (36)$$

Applying Theorem 1 twice, once with γ^i and once with

$(\gamma^i)^2$ in place of y^i , we have that \mathbb{P} -a.s.

$$\lim_{m \rightarrow \infty} \frac{\sum_{i \in I} \gamma^i}{m} = \lim_{m \rightarrow \infty} \frac{\sum_{i \in I \setminus \{0\}} \gamma^i}{m} + \frac{\gamma^0}{m} = \mathbb{E}[\gamma^1] \quad (37)$$

$$\lim_{m \rightarrow \infty} \frac{\sum_{i \in I} (\gamma^i)^2}{m} = \lim_{m \rightarrow \infty} \frac{\sum_{i \in I \setminus \{0\}} (\gamma^i)^2}{m} + \frac{(\gamma^0)^2}{m} = \mathbb{E}[(\gamma^1)^2]$$

However, since $\mathbb{E}[\gamma^1] > 0$ and $\frac{\mathbb{E}[(\gamma^1)^2]}{(\mathbb{E}[\gamma^1])^2} < \infty$ due to the third part of Assumption 2,

$$\lim_{m \rightarrow \infty} \frac{\bar{p} h \sum_{i \in I} (\gamma^i)^2}{\underline{p} m \left(\sum_{i \in I} \gamma^i \right)^2} = 0. \quad (\mathbb{P}\text{-a.s.}) \quad (38)$$

Therefore, since (36) holds for any $\{\gamma^i\}_{i \in I}$, we have that

$$\lim_{m \rightarrow \infty} \frac{F_a^m(x_a)}{F^m(x)} = 0, \quad (\mathbb{P}\text{-a.s.}) \quad (39)$$

Let now $x^*, x_a^* \in X$ denote an optimal solution of \mathcal{P} and \mathcal{P}_a , respectively. By optimality of x_a^* we thus have that

$$F^m(x_a^*) + F_a^m(x_a^*) \leq F^m(x^*) + F_a^m(x^*). \quad (40)$$

Rearranging the terms in (40), and since $F^m(x^*) > 0$ (see discussion above Theorem 2), we obtain

$$\begin{aligned} \frac{F^m(x_a^*) - F^m(x^*)}{F^m(x^*)} &\leq \frac{F_a^m(x^*) - F_a^m(x_a^*)}{F^m(x^*)} \\ &\leq \frac{F_a^m(x^*)}{F^m(x^*)}, \end{aligned} \quad (41)$$

where the last inequality is due to the fact that $F_a^m(x_a^*) \geq 0$. Since (39) holds for any feasible solutions $x, x_a \in X$, it will also hold for $x = x_a = x^*$. Therefore, (39) and (41) lead to

$$\lim_{m \rightarrow \infty} \frac{F^m(x_a^*) - F^m(x^*)}{F^m(x^*)} = 0, \quad (\mathbb{P}\text{-a.s.}) \quad (42)$$

which in turn implies (31), thus concluding the proof. \square

From the proof of Theorem 2 it can be observed that (39) holds even if F, F_a , are evaluated at a possibly different feasible solution of \mathcal{P} and \mathcal{P}_a , respectively. This implies that the auxiliary term included in \mathcal{P}_a tends to be negligible compared to the objective function of \mathcal{P} as the number of agents increases.

The ratio in (31), that asymptotically tends to one, is related to the so called *price of anarchy* in the computer science literature [13], mostly focused on problems where the decision variables are discrete. Informally speaking,

the price of anarchy quantifies the gap between the social optimum and the value of the non-cooperative game; Theorem 2 implies that this gap tends to zero as the number of agents increases, i.e., even if agents act in a non-cooperative manner, as their number increases they tend to a social welfare optimizing behavior. Theorem 2 extends the results of [18] that show asymptotic agreement between Nash equilibria and social optima for the case of homogeneous agents in the absence of constraints, to the more general set-up where agents are subject to heterogenous constraints.

To illustrate the result of Theorem 2, we performed a numerical investigation parametric with respect to the number of agents m . We considered a time horizon $h = 12$, and price coefficients $(p^0, \dots, p^{h-1}) = (0.1, 1, 1.9, 2.8, 3.7, 4.6, 5.5, 6.4, 7.3, 8.2, 9.1, 10)$. For simplicity we assumed that the probability mass is concentrated to the lower and upper limits $\underline{x}^{it} = 0$ and $\bar{x}^{it} = 1$ for all $i \in I \setminus \{0\}$, $t \in H$ (assuming normalized charging rates) that are effectively being treated as deterministic, whereas the charging levels γ^i , $i \in I \setminus \{0\}$, were extracted in an i.i.d. fashion from a uniform distribution with support $[0, 12]$. We consider a zero non-PEV demand and we set the virtual agent 0 accordingly. For any m , we performed 100 multi-extractions of $\{\gamma^i\}_{i \geq 1}$, and calculated the average (across these extractions) of the ratio $\frac{F^m(x_a^*) - F^m(x^*)}{F^m(x^*)}$, where x^*, x_a^* are minimizers of \mathcal{P} and \mathcal{P}_a , respectively. Note that x^*, x_a^* depend on the extracted $\{\gamma^i\}_{i \geq 1}$; however, we suppress this dependence in the notation for simplicity. Figure 1 shows that this ratio tends to zero as the number of agents m increases, as this is expected from (31) (see also (42) in the proof of Theorem 2).

The electricity price in the vehicles' objective functions depends linearly on $\frac{1}{m} \sum_{i \in \mathcal{I}} x^{it}$, therefore, the latter can be thought of as a price incentive that drives the behavior of the vehicles. In Figure 2 we investigate the behavior of this incentive, averaged across the 100 multi-extractions of $\{\gamma^i\}_{i \geq 1}$ and $\underline{x}^{it} = 0$, $\bar{x}^{it} = 1$ for all $i \in I \setminus \{0\}$, $t \in H$, as a function of t . The figure panels correspond to different values of m . The *blue* curve corresponds to the optimal solution of \mathcal{P} ($x = x^*$), whereas the *red* one to the optimal solution of \mathcal{P}_a ($x = x_a^*$). As m increases, the price incentives become identical for all $t \in H$, thus supporting the statement that agents, even if they act non-cooperatively, converge to a social welfare optimizing behavior as their number increases. Note that the price incentives exhibit the same behavior with the corresponding objective functions in Theorem 2, since the latter are strictly convex with respect to these incentives. The non-increasing pattern of the price incentive as a function of time is due to the particular choice for the price coefficients (they increase with respect to t), and it is in general case dependent.

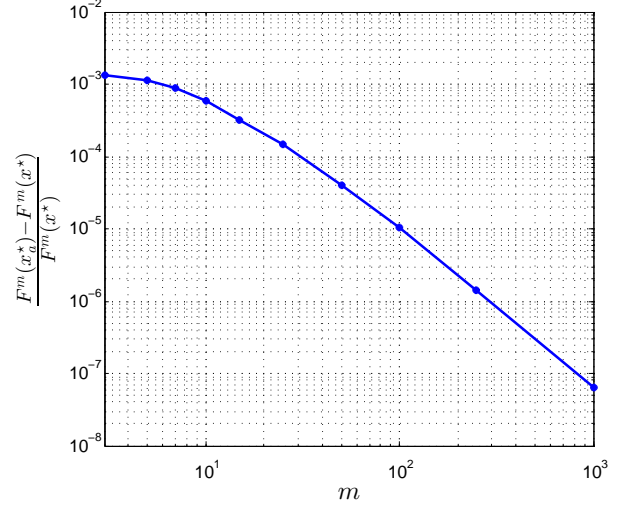


Fig. 1. Relative error $\frac{F^m(x_a^*) - F^m(x^*)}{F^m(x^*)}$, averaged across 100 multi-extractions of $\{\gamma^i\}_{i \geq 1}$ from a uniform distribution and $\underline{x}^{it} = 0$, $\bar{x}^{it} = 1$ for all $i \in I \setminus \{0\}$, $t \in H$, as a function of the number of agents m . The error tends to zero as m increases.

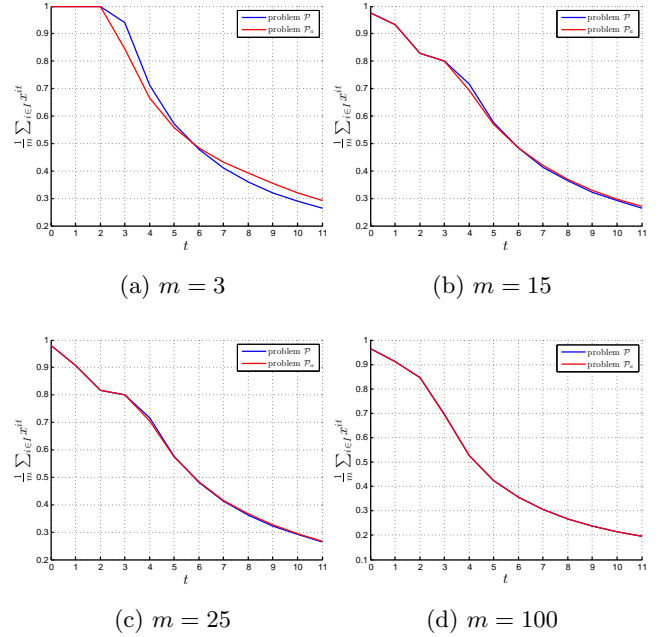


Fig. 2. Price incentive $\frac{1}{m} \sum_{i \in \mathcal{I}} x^{it}$, averaged across 100 multi-extractions of $\{\gamma^i\}_{i \geq 1}$ from a uniform distribution and $\underline{x}^{it} = 0$, $\bar{x}^{it} = 1$ for all $i \in I \setminus \{0\}$, $t \in H$, as a function of t . The figure panels correspond to different values of m . The *blue* curve corresponds to the optimal solution of \mathcal{P} ($x = x^*$), whereas the *red* one to the optimal solution of \mathcal{P}_a ($x = x_a^*$). As m increases (inspect the different figure panels), the price incentives become identical for all $t \in H$.

4 Effect of heterogeneity

To ease notation define the random vectors $\{\xi^i\}_{i \geq 1} = \{\gamma^i, \underline{x}^i, \bar{x}^i\}_{i \geq 1}$. For the results of this section we assume that $\{\xi^i\}_{i \geq 1}$ are extracted from a discrete probability distribution. We thus impose the following assumption.

Assumption 3 Let $\{\xi^i\}_{i \geq 1} = \{\gamma^i, \underline{x}^i, \bar{x}^i\}_{i \geq 1}$ be an infinite sequence of positive, i.i.d. random variables on a discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that \mathbb{P} is supported on n_ξ masses located at $\bar{\xi}^\ell = [\gamma^\ell, \underline{x}^\ell, \bar{x}^\ell]$, $\ell \in L$, where $L = \{1, \dots, n_\xi\}$, i.e., $\sum_{\ell \in L} \mathbb{P}\{\xi = \bar{\xi}^\ell\} = 1$, for any $\xi \in \Omega$.

As it is clear, the setting where all vehicles have the same upper and lower bounds and heterogeneity among agents is only due to the different charging levels γ^i is covered by Assumption 3 as a special case. Note that Assumption 3 is not imposed on the virtual agent 0, which has a deterministic γ^0 , and its own bounds $\underline{x}^{0t} = \bar{x}^{0t}$.

4.1 Abstraction in homogeneous groups

In this subsection we focus on a finite number of agents and show that, either when solving \mathcal{P} or \mathcal{P}_a , the decision vectors corresponding to agents that form a homogeneous group are identical, i.e., identical vehicles have the same charging profile. This naturally provides a way to abstract the overall problem, involving a possibly high number of agents and hence decision vectors, to a problem of smaller size where we only have one decision vector per group of homogeneous agents.

For any $m \geq 1$, for all $i \in I \setminus \{0\}$, denote by $\sum_{i \in I \setminus \{0\}} \mathbb{1}_{\{\xi^i = \bar{\xi}^\ell\}}$ the number of agents that form a homogeneous group with parameter $\bar{\xi}^\ell$, where $\mathbb{1}_{\{\xi^i = \bar{\xi}^\ell\}}$ is an indicator function that is 1 if $\xi^i = \bar{\xi}^\ell$ and 0 otherwise. For all $\ell \in L$, denote by $I^\ell = \{i \in I \setminus \{0\} : \xi^i = \bar{\xi}^\ell\}$ the set of indices corresponding to agents belonging to the same homogeneous group. Note that for the single agent case (i.e., $m = 1$) one of the sets I^ℓ , $\ell \in L$, is singleton and all the others are empty. This implies that there is only one term in the square in (44) below.

Let $\bar{x}^\ell = [\bar{x}^{\ell 0}, \dots, \bar{x}^{\ell(h-1)}]^\top \in \mathbb{R}^{|H|}$, $\ell \in L$, $\bar{x} = (x^0, \bar{x}^1, \dots, \bar{x}^{n_\xi})$, where the variable x^0 is associated to the virtual agent which constitutes a group on its own, $\bar{X} = X^0 \times \bar{X}^1 \times \dots \times \bar{X}^{n_\xi}$, and consider the following variant of \mathcal{P} , where we only consider one decision vector per group of homogeneous agents.

$$\bar{\mathcal{P}} : \min_{\bar{x} \in \bar{X}} \bar{F}^m(\bar{x}), \quad (43)$$

where

$$\bar{F}^m(\bar{x}) = \sum_{t \in H} p^t \left(\sum_{\ell \in L} \sum_{i \in I \setminus \{0\}} \mathbb{1}_{\{\xi^i = \bar{\xi}^\ell\}} \bar{x}^{\ell t} + x^{0t} \right)^2, \quad (44)$$

and for all $\ell \in L$,

$$\bar{X}^\ell = \{\bar{x}^\ell \in \mathbb{R}^{|H|} : \sum_{t \in H} \bar{x}^{\ell t} = \gamma^\ell \text{ and } \bar{x}^{\ell t} \in [\underline{x}^\ell, \bar{x}^\ell], \text{ for all } t \in H\}. \quad (45)$$

Let also $\bar{\mathcal{P}}_a$ denote the variant of \mathcal{P} , defined similarly to $\bar{\mathcal{P}}$ with the difference that its objective function is the sum of the objective function in (43) and the term $\sum_{t \in H} p^t (\bar{x}^{\ell t})^2$ (see also (20)). We then have the following proposition.

Proposition 4 Consider Assumptions 1 and 3. Let $\bar{x}^* \in \bar{X}$, $\bar{x}_a^* \in \bar{X}$ be any minimizer of $\bar{\mathcal{P}}$ and $\bar{\mathcal{P}}_a$, respectively. For all $\ell \in L$, consider the solutions

$$x^{i,*} = \bar{x}^{\ell,*}, \text{ for all } i \in I^\ell, \quad (46)$$

$$x_a^{i,*} = \bar{x}_a^{\ell,*}, \text{ for all } i \in I^\ell, \quad (47)$$

and let $x^* = (x^0, x^{1,*}, \dots, x^{m,*})$ and $x_a^* = (x_a^0, x_a^{1,*}, \dots, x_a^{m,*})$. The solutions x^* , x_a^* are minimizers of \mathcal{P} and \mathcal{P}_a , respectively.

Proof For all $\ell \in L$, for all $i \in I^\ell$, $X^i = \bar{X}^\ell$, and $\bar{X}^0 = X^0$. Therefore, since \bar{x}^* is optimal for $\bar{\mathcal{P}}$, it will be also feasible, i.e., $\bar{x}^{\ell,*} \in \bar{X}^\ell$, for all $\ell \in L$. The last statement, together with (46), leads to $x^{i,*} \in X^i$, for all $i \in I$, which in turn implies that x^* is a feasible solution for \mathcal{P} . Via an analogous argument it can be shown that x_a^* is a feasible solution for \mathcal{P}_a .

By (44) we have that

$$\begin{aligned} \bar{F}^m(\bar{x}^*) &= \sum_{t \in H} p^t \left(\sum_{\ell \in L} \sum_{i \in I \setminus \{0\}} \mathbb{1}_{\{\xi^i = \bar{\xi}^\ell\}} \bar{x}^{\ell t,*} + x^{0t} \right)^2 \\ &= \sum_{t \in H} p^t \left(\sum_{\ell \in L} \sum_{i \in I^\ell} x^{it,*} + x^{0t} \right)^2 \\ &= \sum_{t \in H} p^t \left(\sum_{i \in I \setminus \{0\}} x^{it,*} + x^{0t} \right)^2 = F^m(x^*), \end{aligned} \quad (48)$$

where the third equality is due to (46), and the last one is due to (4).

Let $x = (x^0, \dots, x^m) \in X$ be an arbitrary feasible solution of \mathcal{P} , i.e., $x^i \in X^i$ for all $i \in I$, and consider $\bar{x}^\ell = \frac{1}{n^\ell} \sum_{i \in I^\ell} x^i$, for all $\ell \in L$. Notice that, for $\ell \in L$, since \bar{x}^ℓ is a convex combination of $\{x^i \in X^i\}_{i \in I^\ell}$, $X^i = \bar{X}^\ell$ for

all $i \in I^\ell$ and \bar{X}^ℓ is convex, $\bar{x}^\ell \in \bar{X}^\ell$. We then have that

$$\begin{aligned} \bar{F}^m(\bar{x}^*) &\leq \bar{F}^m(\bar{x}) = \sum_{t \in H} p^t \left(\sum_{\ell \in L} \sum_{i \in I \setminus \{0\}} \mathbb{1}_{\{\xi^i = \bar{\xi}^\ell\}} \bar{x}^{\ell t} + x^{0t} \right)^2 \\ &\leq \sum_{t \in H} p^t \left(\sum_{\ell \in L} \sum_{i \in I^\ell} x^{it} + x^{0t} \right)^2 \\ &= \sum_{t \in H} p^t \left(\sum_{i \in I} x^{it} \right)^2 = F^m(x), \end{aligned} \quad (49)$$

where the first inequality is due to optimality of \bar{x}^* for $\bar{\mathcal{P}}$, whereas the second inequality is due to convexity of \bar{F}^m and the fact that it is quadratic with respect to \bar{x} (see also (44)).

By (48) and (49), we have that $F^m(x^*) \leq F^m(x)$. Since $x \in X$ was arbitrary, we infer that x^* is optimal for \mathcal{P} . To show that x_a^* is optimal for \mathcal{P}_a we follow the same derivation with (48) and (49), appending to \bar{F}^m the term $\sum_{\ell \in L} \sum_{t \in H} p^t (x^{\ell t})^2$. \square

Proposition 4 implies that it suffices to solve $\bar{\mathcal{P}}$ (similarly for \mathcal{P}_a), which involves fewer decision variables compared to \mathcal{P} , and then construct a minimizer of \mathcal{P} by means of the assignment in (46). Note that (46) and (47) enforce the same decision vector to all members of a homogeneous group.

4.2 Asymptotic effect of heterogeneity

In this subsection we investigate the problem of Section 4.1 in the limiting case where the number of agents tends to infinity. In particular, our result shows that as the number of agents increases, the optimal value of \mathcal{P} (normalized as appropriate) tends to a deterministic quantity, which only depends on the distribution of $\{\xi^i\}_{i \geq 1}$ and the solution of $\bar{\mathcal{P}}$. The following theorem provides theoretical support for the intuition that heterogeneity averages out as the number of agents increases.

Theorem 3 *Consider Assumptions 1 and 3. For any $m \geq 1$, denote by $F^m(x^*)$ the optimal value of \mathcal{P} , where x^* is any of its minimizers, and let \bar{x}^* be a minimizer of $\bar{\mathcal{P}}$. We then have that*

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{F^m(x^*)}{m^2} &= \sum_{t \in H} p^t \left(\sum_{\ell \in L} \mathbb{P}\{\xi = \bar{\xi}^\ell\} \bar{x}^{\ell t, *} \right)^2, \quad (\mathbb{P}\text{-a.s.}) \end{aligned} \quad (50)$$

Proof For all $\ell \in L$, by Theorem 1 with $\mathbb{1}_{\{\xi^i = \bar{\xi}^\ell\}}$ in place of y^i , and since $\mathbb{E}[\mathbb{1}_{\{\xi = \bar{\xi}^\ell\}}] = \mathbb{P}\{\xi = \bar{\xi}^\ell\}$ with ξ being a random vector in Ω , we have that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i \in I \setminus \{0\}} \mathbb{1}_{\{\xi^i = \bar{\xi}^\ell\}} = \mathbb{P}\{\xi = \bar{\xi}^\ell\}, \quad (\mathbb{P}\text{-a.s.}) \quad (51)$$

By (48) in Proposition 4 we have that $F^m(x^*) = \bar{F}^m(\bar{x}^*)$, where x^* , \bar{x}^* , are minimizers of \mathcal{P} and $\bar{\mathcal{P}}$, respectively. By the last statement, and the definition of \bar{F}^m in (44) we have that

$$\frac{F^m(x^*)}{m^2} = \sum_{t \in H} p^t \left(\sum_{\ell \in L} \sum_{i \in I \setminus \{0\}} \frac{1}{m} \mathbb{1}_{\{\xi^i = \bar{\xi}^\ell\}} \bar{x}^{\ell t, *} + \frac{x^{0t}}{m} \right)^2, \quad (52)$$

Since (52) holds for any $\{\xi^i\}_{i \in I}$, for any $m \geq 1$, (51) and (52) lead to (50), and hence conclude the proof. \square

Note, that Theorem 2 shows that the ratio between the optimal values of \mathcal{P} and \mathcal{P}_a tends to one as m tends to infinity, for almost any $\{\xi^i\}_{i \geq 1}$, however, their individual values may change for different values of $\{\xi^i\}_{i \geq 1}$. For the case of a discrete probability distribution, Theorem 3 shows that this is not the case and, as the number of agents tends to infinity, the optimal value of \mathcal{P} (and hence also the value of the associated game) tends to a deterministic quantity which depends on the underlying discrete probability distribution. As an effect of Theorem 2 this will also be the case for the optimal value of \mathcal{P}_a , as this tends to the one of \mathcal{P} as the number of agents increases.

The implication of Theorem 3 is illustrated in Figure 3. For different values of m , we provide the empirical probability distribution of $\frac{F^m(x^*)}{m^2}$. We assumed that the probability mass is concentrated to the lower and upper limits $\underline{x}^{it} = 0$ and $\bar{x}^{it} = 1$ for all $i \in I \setminus \{0\}$, $t \in H$, that are effectively being treated as deterministic, whereas the charging levels $\{\gamma^i\}_{i \geq 1}$, $i \in I \setminus \{0\}$, were extracted in an i.i.d. fashion from a discrete uniform distribution in $[0, 12]$, with masses centered uniformly in this interval with spacing 0.01. For any m , we performed 100 multi-extractions of $\{\gamma^i\}_{i \geq 1}$, and for each of them calculated the resulting optimal solution x^* of \mathcal{P} . Numerical values for all other quantities are identical to the ones reported in Section 3.3. It should be noted that as m increases, the empirical distribution becomes concentrated at a single value of $\frac{F^m(x^*)}{m^2}$, in agreement with the statement of Theorem 3. Note that the distribution of the value of the associated game $\frac{F^m(x_a^*)}{m^2}$ is similar with the one of Figure 3; in fact for high values of m it is identical with that of $\frac{F^m(x^*)}{m^2}$ as a result of Theorem 2.

5 Electric vehicle charging control problem revisited

5.1 Decentralized social optimum and Nash equilibrium computation

In the previous sections we studied the properties of the Nash equilibrium of the game formulated in Section 2.2,

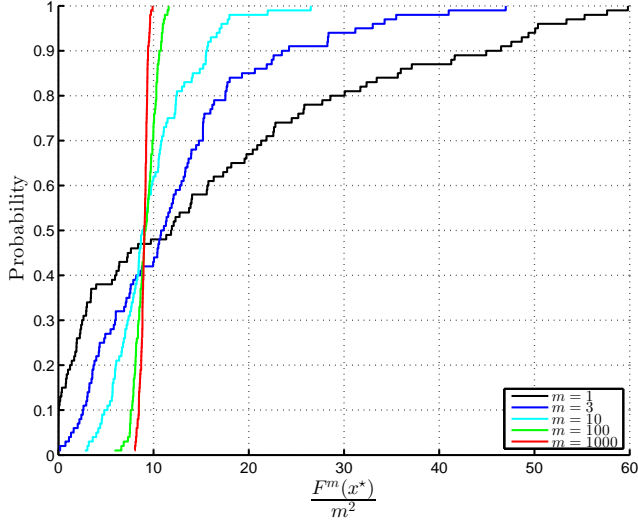


Fig. 3. Empirical distribution of $\frac{F^m(x^*)}{m^2}$, constructed by calculating the optimal solution x^* of \mathcal{P} for 100 multi-extractions of $\{\gamma^i\}_{i \geq 1}$ from a discrete uniform distribution and $\underline{x}^{it} = 0$ and $\bar{x}^{it} = 1$ for all $i \in I \setminus 0$, $t \in H$. As the number of agents m increases the distribution gets concentrated around the quantity in (50).

and its relation with social optima. Note that this equilibrium is unique, as a result of the analysis of Section 3.2 (see also discussion below the proof of Proposition 3). Here we provide a decentralized algorithm to calculate a social optimum of \mathcal{P} and the desired Nash equilibrium.

To achieve this, we exploit the equivalence between Nash equilibria of the problem of interest and social optima of \mathcal{P}_a established in Section 3.2. Therefore, we view the problem of decentralized Nash equilibrium computation as a the problem of determining the minimizer of \mathcal{P}_a (unique since it is a strictly convex program) in a decentralized fashion. To this end, we employ the algorithm developed in [6] for determining the optimal solution of certain optimization programs that encompass \mathcal{P} and \mathcal{P}_a as particular problem classes. This algorithm is motivated by the equivalence between social optima of \mathcal{P}_a , Nash equilibria of the game of Section 2.2 and fixed-points of the mapping \tilde{T} (see Corollary 1 and Proposition 3), and involves an iterative application of \tilde{T} . The main steps of this iterative process are summarized in Algorithm 1.

For $\delta = 0$ we consider the problem of social optimum computation (the objective function of \mathcal{P} appears in the agents local minimization problem in step 9); whereas for $\delta = 1$ we consider the problem of Nash equilibrium computation, which is in turn equivalent to computing the minimizer of \mathcal{P}_a (the objective function of \mathcal{P}_a appears in the agents local minimization problem in step 9).

The interpretation of Algorithm 1 is that, starting from an arbitrary feasible initial estimate (step 6), at every it-

Algorithm 1 Decentralized computation

- 1: **Social optimum – Nash equilibrium**
 - 2: Set $\delta = 0$ for social optimum computation.
 - 3: Set $\delta = 1$ for Nash equilibrium computation.
 - 4: **Initialization**
 - 5: $k = 0$.
 - 6: Consider $x_0^i \in X^i$, for all $i = 1, \dots, m$.
 - 7: **For** $i = 1, \dots, m$ **repeat until convergence**
 - 8: Agent i receives x_k^{-i} from central authority.
 - 9: $x_{k+1}^i = \arg \min_{z^i \in X^i} [F^m(z^i, x_k^{-i}) + \delta f_a(z^i) + c\|z^i - x_k^i\|^2]$.
 - 10: Agent i transmits x_{k+1}^i to central authority.
 - 11: $k \leftarrow k + 1$.
-

eration k each agent i receives from a central authority a tentative estimate for the solution x_k^{-i} of all other agents (step 8). On the basis of this information, an update step is performed, where a new estimate x_{k+1}^i is computed (step 9), by means of a local minimization program. In its local optimization problem, each agent minimizes with respect to its own decision vector while keeping the decision vectors corresponding to the other agents fixed to their values x_k^{-i} at the previous iteration of the algorithm. The corresponding objective function is the sum of the agents' pay-off functions, which is $F^m(\cdot, x_k^{-i})$ or $F^m(\cdot, x_k^{-i}) + f_a(\cdot)$ according to the choice of δ , and a regularization term that penalizes the Euclidean distance of the new estimate x_{k+1}^i from its previous value x_k^i . The relative importance between these two terms is dictated by the regularization coefficient $c > 0$, whose choice as far as convergence of the algorithm is concerned will be discussed in the sequel. Note that step 6 of Algorithm 1 can be rewritten as $x_{k+1}^i = \tilde{T}(x_k)$, and corresponds to the so called Picard-Banach iteration of the mapping \tilde{T} . Note also that steps 8, 9 and 10 of Algorithm 1 are performed in parallel by the agents.

For an appropriate choice of $c > 0$, Algorithm 1 converges, and the returned solution is a minimizer of \mathcal{P} (social optimum) if $\delta = 0$, and the minimizer of \mathcal{P}_a (Nash equilibrium) if $\delta = 1$. This is summarized in the following theorem.

Theorem 4 *Consider Assumptions 1 and 2. For any $m \geq 1$, if $c > m\bar{p}$, then Algorithm 1 converges to a minimizer of \mathcal{P} if $\delta = 0$, and to the minimizer of \mathcal{P}_a if $\delta = 1$.*

Proof Problems \mathcal{P} and \mathcal{P}_a are convex quadratic programs. Denote by $\mathbf{1}_{m+1} \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ a matrix with all entries equal to 1, by Id the identity matrix of appropriate dimension, by \otimes the Kronecker product, and let $P = \text{diag}(p^t)$ be a matrix whose diagonal entries are p^t , $t \in H$. Set then $Q_\delta = (\mathbf{1}_{m+1} + \delta \text{Id}) \otimes P$ and notice that $x^\top Q_\delta x = \sum_{i \in I} [f(x^i, x^{-i}) + \delta f_a(x^i)]$. Therefore, the op-

timization program

$$\min_{x \in X} x^\top Q_\delta x, \quad (53)$$

reduces to \mathcal{P} if $\delta = 0$, and to \mathcal{P}_a if $\delta = 1$. Let $Q_z = (\mathbf{1}_{m+1} - \text{Id}) \otimes P$ be the matrix obtained considering only the off-diagonal entries of Q_δ . By a direct application of Theorem 4 in [6], we have that if $c > \lambda_{Q_z}^{\max}$, where $\lambda_{Q_z}^{\max}$ denotes the maximum eigenvalue of Q_z , then Algorithm 1 converges to an optimizer of (53), and hence to a minimizer of \mathcal{P} or \mathcal{P}_a according to the choice of δ . Note that Q_z , and hence $\lambda_{Q_z}^{\max}$ is independent of δ , and it can be easily shown that $\lambda_{Q_z}^{\max} = m \max_{t \in H} p^t = m\bar{p}$, thus concluding the proof. \square

Note that if Assumption 3 is satisfied, the abstraction of Section 4.1 can be performed. Algorithm 1 is still applicable, with \mathcal{P} and \mathcal{P}_a (and the corresponding objective functions and constraints) being replaced by $\bar{\mathcal{P}}$ and $\bar{\mathcal{P}}_a$, respectively.

5.2 Simulation results

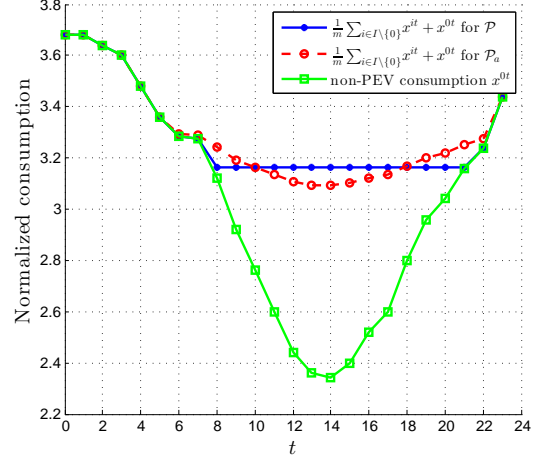
In this section we revisit the PEV charging control problem of Section 2, and perform further numerical investigations by calculating a social optimum and the Nash equilibrium of the game formulated in Section 2.2 in a decentralized fashion via Algorithm 1. We explicitly highlight the presence of non-PEV demand given by x^{0t} , $t \in H$. The resulting optimization program is

$$\begin{aligned} \min_{\{x^i \in X^i\}_{i \in I}} \sum_{t \in H} p^t \left(\sum_{i \in I \setminus \{0\}} x^{it} + x^{0t} \right)^2 \\ + \delta \sum_{t \in H} p^t \sum_{i \in I} (x^{it})^2 \end{aligned} \quad (54)$$

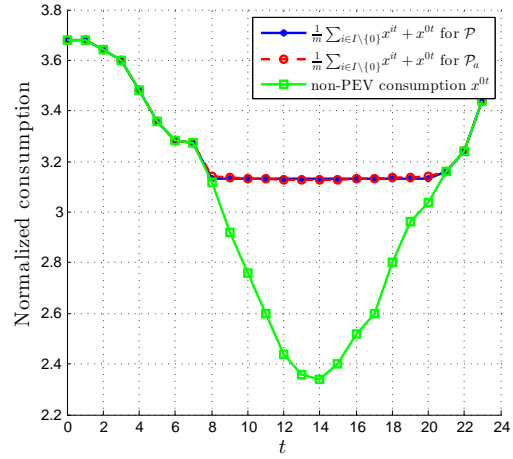
For the $\delta = 0$ we obtain the variant of \mathcal{P} that includes non-PEV consumption, and for $\delta = 1$ the corresponding variant of \mathcal{P}_a .

We employ Algorithm 1 and solve problem (54) for $m = 5$ and $m = 100$. We consider a time horizon $h = 24$, and a fixed price coefficient $p^t = 1$, for all $t \in H$. For all agents $i \in I \setminus \{0\}$ and for all $t \in H$, we set again $\bar{x}^{it} = 1$ and $\underline{x}^{it} = 0$, while the charging levels γ^i , $i \in I \setminus \{0\}$, are extracted in an i.i.d. fashion from a uniform distribution with support $[0, 12]$. The non-PEV consumption profile is retrieved from [18] and is normalized as appropriate to keep it comparable with the PEV consumption.

Figure 4 shows the normalized total consumption profile $\frac{1}{m} \sum_{i \in I \setminus \{0\}} x^{it} + x^{0t}$ obtained by solving problem \mathcal{P} (blue) and problem \mathcal{P}_a (red), by means of Algorithm 1. The non-PEV consumption profile x^{0t} is shown in green. Both solutions have the so called valley filling property, i.e., the PEV consumption tends to compensate for the



(a) $m = 5$



(b) $m = 100$

Fig. 4. Normalized total consumption profile $\frac{1}{m} \sum_{i \in I \setminus \{0\}} x^{it} + x^{0t}$ obtained by solving problem \mathcal{P} (blue) and problem \mathcal{P}_a (red), by means of Algorithm 1. The non-PEV consumption profile x^{0t} is shown in green. As the number of agents increases (inspect the figure panels), the consumption corresponding to the Nash equilibrium solution tends to the social optimum one, i.e., the blue and red lines tend to coincide, in agreement with the result of Theorem 2.

over night drop in the non-PEV consumption. In the case where the number of agents is low ($m = 5$), the social welfare minimizing consumption profile leads to a more flat profile with superior valley filling properties. However, as the number of agents increases ($m = 100$), the consumption corresponding to the Nash equilibrium solution tends to the social optimum one, i.e., the blue and red lines tend to coincide, in agreement with the result of Theorem 2.

6 Concluding remarks

In this paper we considered the problem of optimal charging of Plug-in Electric Vehicles (PEVs) as a multi-agent game. Each vehicle/agent was subject to possibly different constraints, where constraint heterogeneity was represented by assuming that the parameters defining each vehicle constraints are drawn randomly from a given distribution. We showed that, for any finite number of possibly heterogeneous agents, the PEV charging control game admits a unique Nash equilibrium, which is the minimizer of an auxiliary minimization program. At the limiting case of infinite agent populations this Nash equilibrium achieves the same value with the social optimum of the cooperative counterpart of the problem under study for almost any choice of the random heterogeneity parameter. Moreover, in the case where the agents' heterogeneity parameters follow a discrete probability distribution, we showed that agents can be abstracted in homogeneous groups, while the effect of heterogeneity averages out as their number tends to infinity. We provided a decentralized algorithm to compute the desired Nash equilibrium and illustrated the efficacy of our methodology by means of a detailed simulation based study.

Current work concentrates towards extending the developed methodology to a more general class of non-cooperative games, with more involved pay-off functions. Moreover, we aim at extending our approach to the case where agents' constraints are formulated as chance-constraints to be satisfied with a given probability. This can be achieved by exploiting the recent results of [19], where a distributed scenario based methodology for stochastic optimization is proposed.

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